ELSEVIER

Topology and its Applications 79 (1997) 159–172

TOPOLOGY AND ITS APPLICATIONS

Semibundle decompositions of 3-manifolds and the twisted cofundamental group

Louis Zulli [1]

Department of Mathematics, Union College, Schenectady, NY 12308-3166, USA

Received 3 June 1996

Abstract

In this paper we study "semibundle decompositions" of 3-manifolds, that is, decompositions of 3-manifolds as unions of twisted I-bundles over nonorientable surfaces. The primary algebraic tool in this study is a certain easily defined, homotopy-theoretic, abelian group, which we call the "twisted cofundamental group". The main technical work in this paper is the construction of a natural isomorphism between the twisted cofundamental group and a certain one-dimensional cohomology group with "twisted integer" coefficients. We use this isomorphism to determine the structure of the twisted cofundamental group, and we also formulate a "Stallings' fibration theorem" for semibundles. © 1997 Elsevier Science B.V.

Keywords: Semibundle; Twisted cofundamental group; 3-manifold

AMS classification: 57M99

0. Introduction

Informally speaking, a semibundle decomposition of a closed, orientable 3-manifold M is a decomposition $M = M_1 \cup M_2$, where $M_1 \cap M_2 = \Sigma$ is an embedded, orientable surface, and each M_i is a twisted I-bundle over a closed, nonorientable surface, with Σ the corresponding 0-sphere bundle. A 3-manifold M which admits such a decomposition is foliated by compact surfaces; this foliation is induced by a "fibration" $f : M \to D^1$. As the name suggests, a semibundle $f : M \to D^1$ is covered by a surface-bundle $F : M_H \to S^1$, where $Q_H : M_H \to M$ is a two-sheeted covering and $(Q_H)_*\pi_1(M_H, \tilde{*}) = H$ is an index-two subgroup of $\pi_1(M, *)$. In fact, surface-bundle decompositions and semibundle decompositions represent the totality of foliations of 3-manifolds in which all the leaves of the foliation are compact surfaces.

[1] E-mail: zulli@idol.union.edu.

PII S0166-8641(96)00175-7

We begin this article by giving a precise definition of what we mean by a semibundle $f: M \to D^1$, and we then develop the algebraic machinery needed to prove an analogue of Stallings' fibration theorem (see [8]). Indeed, Theorem 5.2 in this paper might be called Stallings' fibration theorem for semibundles. Here is an outline of this work: in Section 1 we define the term "semibundle" and discuss some of the implications of our definition. In Section 2 we introduce the "twisted cofundamental group" of an orbit space X/α, and we explain how semibundles correspond, in a simple way, to elements in this abelian group. In Section 3 we discuss group cohomology—in particular, one-dimensional group cohomology with twisted integer coefficients. In Section 4 we construct a natural isomorphism between the twisted cofundamental group and a certain one-dimensional cohomology group with twisted integer coefficients. This isomorphism is analogous to the isomorphism which represents ordinary one-dimensional integral cohomology as homotopy classes of maps to S^1. (The tools developed in Sections 2–4 are not specific to dimension 3.) Finally, in Section 5, we formulate and prove the aforementioned analogue of Stallings' fibration theorem. Our theorem gives necessary and sufficient algebraic conditions on a twisted cohomology class to guarantee that the class corresponds to a semibundle decomposition of the ambient 3-manifold. In fairness, we should point out that our theorem is not essentially new. Rather, it is a reformulation, in the unifying language of cohomology, of a result proved by Hempel and Jaco [4].

This paper represents an "algebraic" approach to the problem of detecting semibundle decompositions of 3-manifolds. What remains to be explored is the "geometric" analogue of this work—defining a Thurston norm on twisted second homology and using the unit ball in this normed vector space to "locate" semibundle decompositions of the ambient 3-manifold, following the approach taken in [9].

1. Halvings and semibundles

Let M be a closed, connected, orientable 3-manifold. An index-two subgroup of $\pi_1(M, *)$ will be called a *halving* of M. Because M is compact, $\pi_1(M, *)$ is finitely-generated, and thus M has a finite number of halvings—one for each nonzero homomorphism $H_1(M) \to \mathbb{Z}_2$. Corresponding to each halving H of M, there is a two-sheeted covering $Q_H: M_H \to M$, where M_H is a closed, connected, orientable 3-manifold and $(Q_H)_*\pi_1(M_H, \tilde{*}) = H$. Note that the choice of $\tilde{*}$ in $(Q_H)^{-1}(*)$ does not matter, since H is a normal subgroup of $\pi_1(M, *)$. The group of covering translations of the covering $Q_H: M_H \to M$ is of order two; let $\alpha_H: M_H \to M_H$ be a generator. Then α_H is a free involution, and M may be viewed as the orbit space M_H/α_H, with Q_H the canonical quotient map.

Thus, to each halving H of M, we can associate a pair (M_H, α_H). Of course, this pair is not canonically determined by M and H, but it is determined up to equivariant homeomorphism. That is, if (N_H, β_H) were another such pair, there would be a homeomorphism $h: M_H \to N_H$ satisfying $h \circ \alpha_H = \beta_H \circ h$. The existence of a homeomorphism $h: M_H \to N_H$ follows from covering space theory, since M_H and N_H correspond to the same subgroup of $\pi_1(M,*)$, namely H. The equivariance of h—which implies the equivariance of h^{-1}, since α_H and β_H are involutions—follows from the fact that the involutions α_H and β_H are without fixed-points.

For the sake of convenience (and because there is no ambiguity up to equivariant homeomorphism) let us fix, for each halving H of M, a specific double-covering $Q_H : M_H \to M$, and hence a specific free involution $\alpha_H : M_H \to M_H$. That is, for each halving H of M, let us simply decree that M is to be viewed as the orbit space M_H/α_H, for a specific 3-manifold M_H and a specific involution α_H.

We can now define our primary objects of study. Let M be as above, and let H be a halving of M. Let (M_H, α_H) be the associated double-cover and involution. View S^1 as the unit circle in $\mathbb{C}$ and D^1 as $[-1,1]$ in $\mathbb{R}$. Let $q : S^1 \to D^1$ be the projection $z \mapsto \mathrm{Re}(z)$, and let $\tau : S^1 \to S^1$ denote complex-conjugation. Let Σ be a closed, connected, orientable surface. A map $f : M \to D^1$ will be called a *semibundle subordinate to the halving H* (or, more briefly, an *H-semibundle*) *with regular fiber Σ* if there exists a fiber-bundle $F : M_H \to S^1$ with fiber Σ which satisfies $q \circ F = f \circ Q_H$ and $\tau \circ F = F \circ \alpha_H$.

Before we discuss the implications of this definition, two comments are in order.

First, we could have allowed the fiber surface Σ to be disconnected. This is the situation which would arise were we to compose a semibundle $f : M \to D^1$ (as defined above) with the "k-fold folding map", the map $D^1 \to D^1$ covered by $z^k : S^1 \to S^1$. In fact, one can show that any "semibundle with disconnected fiber" arises, up to homeomorphism, in precisely this manner. Having said this, we will restrict our attention to the case where Σ is connected.

Secondly, it is important to note that, in our definition of semibundle, the map F covering a given semibundle $f : M \to D^1$ is not unique. In fact, it is not difficult to show that there are exactly two such maps, namely F and $\tau \circ F$.

Let us now consider what our definition of semibundle entails. First note that a semibundle $f : M \to D^1$ is necessarily a surjection, since F and q are surjections. For $t = 1$ and for $t = -1$,

$$(Q_H)^{-1} f^{-1}(t) = F^{-1}(t)$$

is a copy of the surface Σ. From the equivariance condition $\tau \circ F \doteq F \circ \alpha_H$, it follows that each of these copies of Σ is carried to itself by the restriction of α_H. Hence, for $t = 1$ and for $t = -1$, $f^{-1}(t) \subseteq M$ is an embedded surface having $F^{-1}(t) \approx \Sigma$ as a double-cover. For $t \in (-1,1)$,

$$(Q_H)^{-1} f^{-1}(t) = F^{-1} q^{-1}(t)$$

consists of two copies of the fiber Σ, which, by the equivariance condition once more, are interchanged by the restriction of α_H. Thus, for $t \in (-1,1)$, $f^{-1}(t) \subseteq M$ is homeomorphic to Σ. Each of $f^{-1}(J)$, where $J = [-1,0]$ or $J = [0,1]$, is an I-bundle covered by the product I-bundle $(Q_H)^{-1} f^{-1}(J)$. Since their boundaries are the connected surface $f^{-1}(0)$, they are twisted I-bundles, and since M is orientable their base surfaces must be nonorientable. Thus the total space of a semibundle $f : M \to D^1$ with regular fiber

Σ is a union, along the embedded, orientable surface $f^{-1}(0) \approx \Sigma$, of a pair of twisted I-bundles over nonorientable surfaces.

2. The twisted cofundamental group

In this section we define an abelian group, $\overline{\pi}^1(X/\alpha)$, associated to an orbit space X/α, where X is a topological space and $\alpha : X \to X$ is an involution. Semibundles $f : M \to D^1$ subordinate to the halving H of M will correspond, in a simple way, to elements in $\overline{\pi}^1(M_H/\alpha_H)$.

As before, view S^1 as the set of unit-length complex numbers. Then S^1 becomes an abelian group under complex multiplication. Even though S^1 is abelian, we will utilize multiplicative notation throughout. Thus the "zero" in S^1 is the complex number 1, and the "negative" of $z \in S^1$ is the complex number $\overline{z}$. Because both complex multiplication $\mu : S^1 \times S^1 \to S^1$ and complex-conjugation $\tau : S^1 \to S^1$ are continuous, S^1 is an abelian topological group.

Let X be a topological space. The collection of all maps $X \to S^1$ will be denoted $\mathrm{Map}(X, S^1)$. Since S^1 is an abelian topological group, $\mathrm{Map}(X, S^1)$ becomes an abelian group under pointwise-multiplication of maps. The "zero" in $\mathrm{Map}(X, S^1)$ is the constant map at $1 \in S^1$, which will be written simply as 1. The "negative" of a map $f : X \to S^1$ is the map $\tau \circ f$. Note that $\mathrm{Map}(-, S^1)$ is actually a contravariant functor from the category of topological spaces to the category of abelian groups. In particular, a map $h : Y \to X$ induces a homomorphism $h^{\#} : \mathrm{Map}(X, S^1) \to \mathrm{Map}(Y, S^1)$ defined by $h^{\#}(f) = f \circ h$.

Let $[1]$ denote the collection of all maps $X \to S^1$ which are homotopic to the constant map 1, i.e., $[1]$ is the collection of all null-homotopic maps $X \to S^1$. It is easy to verify that $[1]$ is a subgroup of $\mathrm{Map}(X, S^1)$. The quotient group $\mathrm{Map}(X, S^1)/[1]$ is usually denoted $[X, S^1]$; it is the set of homotopy classes of maps $X \to S^1$, with "addition" defined by $[f] \cdot [g] = [f \cdot g]$. If X is a connected CW-complex, then

$$[X, S^1] \approx H^1(X; \mathbb{Z}) \approx \mathrm{Hom}\left(H_1(X), \mathbb{Z}\right) \approx \mathrm{Hom}\left(\pi_1(X, *), \mathbb{Z}\right).$$

For this reason, we will refer to $[X, S^1]$ as the *cofundamental group of* X, and adopt the notation $\pi^1(X)$. Note that $\pi^1(X)$ is a free abelian group of finite rank if $H_1(X)$ is finitely-generated. For more on cohomotopy groups, see [1,5,7].

Suppose now that X is a topological space, and that $\alpha : X \to X$ is an involution. We do not require that α be free. A map $f : X \to S^1$ will be called *trivially equivariant* if $f \circ \alpha = f$; it will be called *nontrivially equivariant* if $f \circ \alpha = \tau \circ f$. The collection of all trivially equivariant maps $X \to S^1$ will be denoted E^+, while the collection of all nontrivially equivariant maps will be denoted E^-. Of course, E^+ and E^- depend on both X and the involution α, but for simplicity this is suppressed from the notation. It is easy to verify that both E^+ and E^- are subgroups of $\mathrm{Map}(X, S^1)$.

Let $[1]^+$ denote the set of all trivially equivariant maps $X \to S^1$ which are homotopic to the constant map 1 through trivially equivariant maps. Let $[1]^-$ denote the set of all nontrivially equivariant maps $X \to S^1$ which are homotopic to the constant map 1 through nontrivially equivariant maps. Note that $[1]^+ \subseteq E^+ \cap [1]$ and $[1]^- \subsetneq E^- \cap [1]$, and that both $[1]^+$ and $[1]^-$ are subgroups of $\mathrm{Map}(X, S^1)$.

Proposition 2.1. $E^+/[1]^+ \approx \pi^1(X/\alpha)$.

Proof. Recall that $\pi^1(X/\alpha) = \mathrm{Map}(X/\alpha, S^1)/[1]$. Let $p: X \to X/\alpha$ denote the canonical projection onto the quotient. Then $p^\# : \mathrm{Map}(X/\alpha, S^1) \to \mathrm{Map}(X, S^1)$ is an isomorphism onto its image $E^+ \subseteq \mathrm{Map}(X, S^1)$. Since $p^\#$ maps $[1]$ onto $[1]^+$, the induced map $p^* : \mathrm{Map}(X/\alpha, S^1)/[1] \to E^+/[1]^+$ is an isomorphism as well. $\square$

Since $E^+/[1]^+$ is canonically isomorphic to the "ordinary" cofundamental group of X/α, it is reasonable to call $E^-/[1]^-$ the *twisted cofundamental group of* X/α. We will denote this group $\bar{\pi}^1(X/\alpha)$. In Section 4 of this article we show that this group is naturally isomorphic—at least if X is a CW-complex—to a certain cohomology group with twisted integer coefficients. The structure of the twisted cohomology group is determined in Section 3.

If $f: M \to D^1$ is an H-semibundle, then f is covered by a pair of nontrivially equivariant surface-bundles, $F: M_H \to S^1$ and $\tau \circ F: M_H \to S^1$. Thus, to an H-semibundle $f: M \to D^1$, we can associate the elements $[F]$ and $[\tau \circ F] = -[F]$ in the twisted cofundamental group $\bar{\pi}^1(M_H/\alpha_H)$. (Since an equivariant homeomorphism $h: (X, \alpha) \to (Y, \beta)$ induces an isomorphism $h^* : \bar{\pi}^1(Y/\beta) \to \bar{\pi}^1(X/\alpha)$, the group $\bar{\pi}^1(M_H/\alpha_H)$ is determined, up to isomorphism, by M and H.)

3. Twisted cohomology

In this section we discuss group cohomology. Because we are only interested in one-dimensional cohomology, we restrict our attention to that special case. This also allows us to utilize an easily-applied classical formulation of group cohomology, which we now review.

Let Π be a group, and let A be an abelian group. Let $\theta: \Pi \to \mathrm{Aut}(A)$ be a homomorphism. Define an action of Π on A by $\gamma \cdot a = \{\theta(\gamma)\}(a)$. A function $f: \Pi \to A$ which satisfies $f(\gamma\eta) = f(\gamma) + \gamma \cdot f(\eta)$ for all $\gamma, \eta \in \Pi$ is called a *crossed homomorphism*. Let $Q(\Pi, A)$ denote the set of all crossed homomorphisms from Π to A. $Q(\Pi, A)$ becomes an abelian group under pointwise-addition of functions.

For each $a \in A$, one can define a function $p_a: \Pi \to A$ by $p_a(\gamma) = a - \gamma \cdot a$. One can verify directly that each such p_a is, in fact, a crossed homomorphism from Π to A. Let $P(\Pi, A) = \{p_a : a \in A\}$ denote the set of these so-called *principal homomorphisms*. It is not difficult to verify that $P(\Pi, A)$ is a subgroup of $Q(\Pi, A)$, and that the assignment $a \mapsto p_a$ is a surjective homomorphism from A onto $P(\Pi, A)$.

The quotient group $Q(\Pi, A)/P(\Pi, A)$ is usually denoted $H^1(\Pi; A)$ and called the *first cohomology group of* Π *with coefficients in* A. Note that $H^1(\Pi; A)$ depends on Π, A and the action of Π on A, although reference to the action is suppressed from the notation. $H^1(-; A)$ is a functor from an appropriately defined category to the category of abelian groups—in particular, a homomorphism $q:\widetilde{\Pi}\to\Pi$ induces a homomorphism $q^*:H^1(\Pi;A)\to H^1(\widetilde{\Pi};A)$ via composition. (The action of $\widetilde{\Pi}$ on A is the pullback, via q, of the action of Π on A.)

Note that, in the case where Π acts trivially on A, $Q(\Pi,A)=\mathrm{Hom}(\Pi,A)$ and $P(\Pi,A)=0$, so that $H^1(\Pi;A)=\mathrm{Hom}(\Pi,A)$. We will be interested in the case where Π acts on A in the simplest nontrivial manner. Namely, let $H\subsetneq\Pi$ be an index-two subgroup, and define an action of Π on A by declaring that elements of H act trivially on A, while elements of $\Pi-H$ act on A by negation. Thus "half" of Π acts trivially on A. For this reason, we will call such an action a *semitrivial action of* (Π,H) *on* A. Since, for any action of Π on A, $\{\gamma\in\Pi\colon\ \gamma\cdot a=a\ \forall a\in A\}=\ker\theta$ is a subgroup of Π, a semitrivial action of (Π,H) on A is as close as one can come to a trivial action of Π on A without actually having one.

For the remainder of this section we will suppose that (Π,H) acts semitrivially on A. In this case, the group $H^1(\Pi;A)$ is sometimes called the *first cohomology group of* Π *with twisted* A *coefficients*. We now determine the structure of this group.

For each $a\in A$, define $\chi_a:\Pi\to A$ by

$$\chi_a(\gamma)=\begin{cases}0 & \text{if } \gamma\in H,\\ a & \text{if } \gamma\notin H,\end{cases}$$

and let $\chi(\Pi,A)=\{\chi_a\colon\ a\in A\}$. By examining cases, it is easy to verify that each χ_a is a crossed homomorphism. It is also easy to see that $\chi(\Pi,A)$ is a subgroup of $Q(\Pi,A)$, and that the assignment $a\mapsto\chi_a$ defines an isomorphism $A\to\chi(\Pi,A)$.

Recall that, for each $a\in A$, there is a principal homomorphism $p_a:\Pi\to A$ defined by $p_a(\gamma)=a-\gamma\cdot a$, and that $p:A\to P(\Pi,A)$ defined by $p(a)=p_a$ is a surjective homomorphism. If (Π,H) acts semitrivially on A, then $p_a=\chi_{2a}$ for each $a\in A$. In particular, $P(\Pi,A)$ is a subgroup of $\chi(\Pi,A)$, and it is easy to check that the isomorphism $a\mapsto\chi_a$ induces an isomorphism $A/2A\to\chi(\Pi,A)/P(\Pi,A)$.

We are interested in the structure of $H^1(\Pi;A)$. To this end, we have

Proposition 3.1. *Let* (Π,H) *act semitrivially on* A. *Then there is a homomorphism* $i^*:H^1(\Pi;A)\to\mathrm{Hom}(H,A)$ *with* $\ker i^*=\chi(\Pi,A)/P(\Pi,A)$.

Proof. Let $i:H\hookrightarrow\Pi$ be the inclusion, and let $i^*:H^1(\Pi;A)\to H^1(H;A)$ be the induced homomorphism. Note that the action of H on A is trivial, being the restriction to H of the semitrivial action of (Π,H) on A. Thus $H^1(H;A)=\mathrm{Hom}(H,A)=Q(H,A)$. Recall that i^* is induced by the homomorphism $i^\#:Q(\Pi,A)\to Q(H,A)$, where $i^\#(f)$ is just the restriction $f\circ i$. The canonical projection $Q(\Pi,A)\to H^1(\Pi;A)$ restricts to a surjection $\ker i^\#\to\ker i^*$ with kernel $P(\Pi,A)$. Thus Proposition 3.1 will follow immediately from

Lemma 3.2. $\ker i^\#=\chi(\Pi,A)$.

Proof. Clearly $\chi(\Pi,A)\subseteq\ker i^\#$, since each χ_a restricts to the zero homomorphism on H. For the other inclusion, suppose that $f\in Q(\Pi,A)$ and that $i^\#(f)=0$, so that f

vanishes on H. We need to show that the restriction of f to $\Pi - H$ is a constant function. Let η and η' be arbitrary elements in $\Pi - H$. Since $\eta' = (\eta'\eta^{-1})\eta$ and $\eta'\eta^{-1} \in H$, we have

$$f(\eta') = f\big((\eta'\eta^{-1})\eta\big) = f(\eta'\eta^{-1}) + f(\eta) = 0 + f(\eta) = f(\eta),$$

so f is indeed constant on $\Pi - H$. □

Corollary 3.3. *Let Π be a finitely-generated group and let (Π, H) act semitrivially on $\mathbb{Z}$. Then $H^1(\Pi;\mathbb{Z})$ is a finitely-generated abelian group whose torsion subgroup is isomorphic to $\mathbb{Z}_2$.*

Proof. By Proposition 3.1, there is a homomorphism $i^* : H^1(\Pi;\mathbb{Z}) \to \mathrm{Hom}(H,\mathbb{Z})$ with $\ker i^* \approx \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$. Since H has finite index in Π, H is finitely-generated. Thus $\mathrm{Hom}(H,\mathbb{Z})$ is a free abelian group of finite rank, and so is the subgroup $i^*(H^1(\Pi;\mathbb{Z}))$. The result follows. □

4. A natural isomorphism

In this section we prove that the twisted cofundamental group $\bar{\pi}^1(X)$ of a connected CW-complex $X = X_H/\alpha_H$ is naturally isomorphic to the twisted cohomology group $H^1(\pi_1(X,*);\mathbb{Z})$, where $(\pi_1(X,*), H)$ acts semitrivially on $\mathbb{Z}$. Thus one-dimensional cohomology with twisted integer coefficients may be represented as nontrivially equivariant homotopy classes of nontrivially equivariant maps to S^1.

Before we can proceed, we need to say a little bit about CW-complexes in our equivariant setting. Let $\widetilde{X}$ be a topological space, and let $\alpha : \widetilde{X} \to \widetilde{X}$ be a free involution. A CW-decomposition $\{f_j\colon j \in J\}$ of $\widetilde{X}$ will be called *α-invariant* if, for each open cell f_j in the decomposition, the open cell $\alpha(f_j)$ is also a cell in the decomposition, and, furthermore, if for each open cell f_j in the decomposition, there exists a characteristic map $\psi_j : D^{m_j} \to \widetilde{X}$ for f_j such that $\alpha \circ \psi_j$ is a characteristic map for $\alpha(f_j)$. If $\widetilde{X}$ admits an α-invariant CW-decomposition, we will call $\widetilde{X}$ an *α-invariant CW-complex.*

Such α-invariant CW-complexes arise naturally when one considers two-sheeted coverings of CW-complexes. Indeed, let $Q : \widetilde{X} \to X$ be a two-sheeted covering of a CW-complex X, and let $\alpha : \widetilde{X} \to \widetilde{X}$ be the free involution generating the group of covering transformations of the covering. Then one can lift the CW-structure on X to an α-invariant CW-structure on $\widetilde{X}$. In what follows, we will always assume that a two-sheeted covering space of a CW-complex is endowed with such a CW-structure.

We will make frequent use of the following "equivariant homotopy-extension" property of α-invariant CW-complexes. The proof of Proposition 4.1 amounts to nothing more than verifying that the usual proof of the homotopy-extension property of CW-complexes may be modified to work in our equivariant setting. We leave such a verification to the reader, or see [6].

Proposition 4.1. *Suppose Y is a topological space, and $\beta : Y \to Y$ is an involution, which may have fixed-points. Suppose A is an α-invariant subcomplex of an α-invariant CW-complex X. Then any equivariant map $h : (X \times \{0\}) \cup (A \times I) \to Y$ extends to an equivariant map $H : X \times I \to Y$. Here "equivariant" means $H(\alpha(x), t) = \beta(H(x,t))$ and $h(\alpha(x), t) = \beta(h(x,t))$.*

Before we proceed, let us clarify our notation. X is a connected CW-complex and H is a halving of X, that is, H is an index-two subgroup of $\pi_1(X, *)$. $Q_H : X_H \to X$ is a two-sheeted covering with $(Q_H)_* \pi_1(X_H, \tilde{*}) = H$, and $\alpha_H : X_H \to X_H$ is the free involution generating the group of covering translations of the covering. X_H is endowed with the α-invariant CW-structure lifted from X. Our aim is to prove

Theorem 4.2. *$\bar{\pi}^1(X)$ is naturally isomorphic to $H^1(\pi_1(X, *); \mathbb{Z})$, where $X = X_H/\alpha_H$ and $(\pi_1(X, *), H)$ acts semitrivially on $\mathbb{Z}$.*

Theorem 4.2 follows immediately from six lemmas.

Lemma 4.3. *$\bar{\pi}^1(X) = E^-/[1]^-$ is naturally isomorphic to $E_1^-/(E_1^- \cap [1]^-)$, where E_1^- is the subgroup of E^- consisting of nontrivially equivariant maps $F : (X_H, \tilde{*}) \to (S^1, 1)$, i.e., basepoint-preserving nontrivially equivariant maps.*

Proof. The inclusion $E_1^- \hookrightarrow E^-$ induces the isomorphism. The key observation is the fact that a nontrivially equivariant map $F : X_H \to S^1$ may be equivariantly homotoped to become basepoint-preserving. This is a consequence of the path-connectedness of S^1 and Proposition 4.1. $\square$

Lemma 4.4. *A nontrivially equivariant map $F : (X_H, \tilde{*}) \to (S^1, 1)$ determines a crossed homomorphism $F_* : \pi_1(X, *) \to \mathbb{Z}$.*

Proof. Let $\Gamma : I \to S^1$ be defined by $\Gamma(t) = \mathrm{e}^{2\pi i t}$, and view $\pi_1(S^1, 1)$ as the free multiplicative group generated by $[\Gamma]$. Let $\gamma : I \to X$ be a loop based at $*$. Let $\widetilde{\gamma} : I \to X_H$ be the unique lift of γ based at $\tilde{*}$. Then $\widetilde{\gamma}(1) = \tilde{*}$ or $\widetilde{\gamma}(1) = \alpha_H(\tilde{*})$, but in either case $F \circ \widetilde{\gamma} : I \to S^1$ is a loop based at 1. Thus $[F \circ \widetilde{\gamma}]$ is an element of $\pi_1(S^1, 1)$, so $[F \circ \widetilde{\gamma}] = [\Gamma]^m$ for a unique integer m. One checks that, if γ is homotopic to γ' rel $\{0,1\}$, then $F \circ \widetilde{\gamma}$ is homotopic to $F \circ \widetilde{\gamma}'$ rel $\{0,1\}$. Thus the integer m depends only on $[\gamma] \in \pi_1(X, *)$, and we define $F_*([\gamma]) = m$.

We must show that $F_* : \pi_1(X, *) \to \mathbb{Z}$ is a crossed homomorphism. To fix notation, let γ and η be loops in X based at $*$, and let σ be the concatenation $\gamma\eta$. Thus $[\sigma] = [\gamma][\eta]$ in $\pi_1(X, *)$. We need to show that $F_*([\sigma]) = F_*([\gamma]) + \gamma \cdot F_*([\eta])$. There are two cases:

Case 1: $[\gamma] \in H$. In this case γ lifts to a loop in X_H based at $\tilde{*}$, so $\tilde{\sigma} = \widetilde{\gamma}\tilde{\eta}$. Then

$$\begin{aligned}[F \circ \tilde{\sigma}] &= [F \circ (\widetilde{\gamma}\tilde{\eta})] = [(F \circ \widetilde{\gamma})(F \circ \tilde{\eta})] = [F \circ \widetilde{\gamma}][F \circ \tilde{\eta}] \\ &= [\Gamma]^{F_*([\gamma])}[\Gamma]^{F_*([\eta])} = [\Gamma]^{F_*([\gamma]) + F_*([\eta])}.\end{aligned}$$

Thus $F_*([\sigma]) = F_*([\gamma]) + F_*([\eta]) = F_*([\gamma]) + \gamma \cdot F_*([\eta])$, since $[\gamma]$ acts trivially in this case.

Case 2: $[\gamma] \notin H$. In this case γ lifts to a path in X_H from $\tilde{*}$ to $\alpha_H(\tilde{*})$, so $\tilde{\sigma} = \tilde{\gamma}(\alpha_H \circ \tilde{\eta})$. Then

$$\begin{aligned}[F \circ \tilde{\sigma}] &= \big[F \circ \big(\tilde{\gamma}(\alpha_H \circ \tilde{\eta})\big)\big] = \big[(F \circ \tilde{\gamma})\big(F \circ (\alpha_H \circ \tilde{\eta})\big)\big] = [F \circ \tilde{\gamma}]\big[(F \circ \alpha_H) \circ \tilde{\eta}\big] \\ &= [F \circ \tilde{\gamma}]\big[(\tau \circ F) \circ \tilde{\eta}\big] = [F \circ \tilde{\gamma}]\tau_*([F \circ \tilde{\eta}]) \\ &= [F \circ \tilde{\gamma}][F \circ \tilde{\eta}]^{-1} \quad (\text{since } \tau_* : \pi_1(S^1, 1) \to \pi_1(S^1, 1) \text{ carries } [\Gamma] \text{ to } [\Gamma]^{-1}) \\ &= [\Gamma]^{F_*([\gamma])}\big([\Gamma]^{F_*([\eta])}\big)^{-1} = [\Gamma]^{F_*([\gamma]) - F_*([\eta])}.\end{aligned}$$

Thus $F_*([\sigma]) = F_*([\gamma]) - F_*([\eta]) = F_*([\gamma]) + \gamma \cdot F_*([\eta])$, since $[\gamma]$ acts nontrivially in this case.

This proves Lemma 4.4. □

Lemma 4.5. *The assignment $F \mapsto F_*$ is a homomorphism $E_1^- \to Q(\pi_1(X, *), \mathbb{Z})$.*

Proof. Let $F, G : (X_H, \tilde{*}) \to (S^1, 1)$ be nontrivially equivariant maps. Let $\gamma : I \to X$ be a loop based at $*$, and let $\tilde{\gamma} : I \to X_H$ be the unique lift of γ based at $\tilde{*}$. Then $(F \cdot G) \circ \tilde{\gamma} = (F \circ \tilde{\gamma}) \cdot (G \circ \tilde{\gamma})$. Now $F \circ \tilde{\gamma}$ is homotopic to $\Gamma^{F_*([\gamma])}$ rel $\{0, 1\}$, while $G \circ \tilde{\gamma}$ is homotopic to $\Gamma^{G_*([\gamma])}$ rel $\{0, 1\}$. Pointwise-multiplication of these homotopies yields a homotopy rel $\{0, 1\}$ from $(F \circ \tilde{\gamma}) \cdot (G \circ \tilde{\gamma})$ to $\Gamma^{F_*([\gamma])} \cdot \Gamma^{G_*([\gamma])} = \Gamma^{(F_* + G_*)([\gamma])}$. Thus $(F \cdot G)_*([\gamma]) = (F_* + G_*)([\gamma])$, and the lemma is proved. □

Lemma 4.6. *The assignment $F \mapsto F_*$ is a surjection $E_1^- \to Q(\pi_1(X, *), \mathbb{Z})$.*

Proof. Let $f : \pi_1(X, *) \to \mathbb{Z}$ be a crossed homomorphism. We define a nontrivially equivariant map $F : (X_H, \tilde{*}) \to (S^1, 1)$ inductively over the skeleta of X_H as follows:

0-skeleton. Define $F(\tilde{*}) = F(\alpha_H(\tilde{*})) = 1$.

1-skeleton. For each 1-cell e_i^1 in X, let $a_i : (I, 0, 1) \to (X^{(1)}, *, *)$ be a characteristic map. Let $\tilde{a}_i : I \to X_H^{(1)}$ be the unique lift of a_i based at $\tilde{*}$. Note that $\tilde{a}_i$ is a characteristic map for a 1-cell $\tilde{e}_i^1$ covering e_i^1, and that $\alpha_H \circ \tilde{a}_i$ is a characteristic map for the other 1-cell covering e_i^1, namely $\alpha_H(\tilde{e}_i^1)$. To define F on $\tilde{e}_i^1$, it suffices to define a map $F_i' : (I, 0, 1) \to (S^1, 1, 1)$. So let $F_i' = \Gamma^{f([a_i])}$. That is, F should wrap $\tilde{e}_i^1$ $f([a_i])$ times counter-clockwise around S^1 from 1 back to 1. Define F on $\alpha_H(\tilde{e}_i^1)$ to be the composition $\tau \circ F \circ \alpha_H$. Do this for all 1-cells e_i^1 in X.

At this point we have a nontrivially equivariant map $F : (X_H^{(1)}, \tilde{*}) \to (S^1, 1)$ which we wish to extend over all of X_H. Applying the construction of Lemma 4.4 to this map, we obtain a crossed homomorphism $F_* : \pi_1(X^{(1)}, *) \to \mathbb{Z}$. (Here $[\gamma] \in \pi_1(X^{(1)}, *)$ acts on $m \in \mathbb{Z}$ by $[\gamma] \cdot m = i_*([\gamma]) \cdot m$, where $i_* : \pi_1(X^{(1)}, *) \to \pi_1(X, *)$ is the map induced by inclusion.) One now verifies that $f \circ i_* : \pi_1(X^{(1)}, *) \to \mathbb{Z}$ is also a crossed homomorphism, and that, by construction, $f \circ i_*$ agrees with F_* on the generators $[a_i]$ of $\pi_1(X^{(1)}, *)$. Because they agree on generators, the crossed homomorphisms $f \circ i_*$ and F_* are equal.

We can now complete the extension of F over X_H.

2-skeleton. Let e_i^2 be a 2-cell in X, attached by the loop $\phi:(I,0,1) \to (X^{(1)},*,*)$. Let $\tilde{\phi}:(I,0,1) \to (X_H^{(1)},\tilde{*},\tilde{*})$ be the unique lift of ϕ based at $\tilde{*}$. Note that $\tilde{\phi}$ is necessarily a loop, since ϕ is null-homotopic in X. Indeed, $\tilde{\phi}$ is the attaching map of a 2-cell $\widetilde{e}_i^2$ covering e_i^2, and $\alpha_H \circ \tilde{\phi}$ is an attaching map for the other 2-cell covering e_i^2, namely $\alpha_H(\widetilde{e}_i^2)$. To show that we can extend F over $\widetilde{e}_i^2$, it suffices to show that $F \circ \tilde{\phi}$ is null-homotopic in S^1. By the remarks above, $F_*([\phi]) = f(i_*([\phi])) = f(1) = 0$ (a crossed homomorphism takes the identity in the group to 0 in the abelian coefficient group), and so $[F \circ \tilde{\phi}] = [\Gamma]^0$, meaning that $F \circ \tilde{\phi}$ is null-homotopic in S^1. Thus we can extend F over $\widetilde{e}_i^2$. Define F on $\alpha_H(\widetilde{e}_i^2)$ to be the composition $\tau \circ F \circ \alpha_H$. Do this for all 2-cells e_i^2 in X.

3-skeleton. Let e_i^3 be a 3-cell in X. Let $\widetilde{e}_i^3$ be either of the 3-cells in X_H which cover e_i^3. Let $\tilde{\phi}:(S^2,*) \to (X_H^{(2)},\tilde{\phi}(*))$ be an attaching map for $\widetilde{e}_i^3$. Now $F\circ\tilde{\phi}$ represents an element in $\pi_2(S^1,*)$; since this group is trivial, $F \circ \tilde{\phi}$ is null-homotopic. Thus we can extend F over $\widetilde{e}_i^3$. As before, define F on $\alpha_H(\widetilde{e}_i^3)$ to be $\tau \circ F \circ \alpha_H$. Do this for all 3-cells e_i^3 in X.

Higher skeleta. Because S^1 is aspherical, we can extend F over all the higher skeleta of X_H in the same manner as for the 3-skeleton.

Thus we obtain a nontrivially equivariant map $F:(X_H,\tilde{*}) \to (S^1,1)$. Because the crossed homomorphism $F_*:\pi_1(X,*) \to \mathbb{Z}$ agrees with f on generators of $\pi_1(X,*)$, $F_* = f$ and the lemma is proved. □

At this point we have a surjection $\Phi: E_1^- \to Q(\pi_1(X,*),\mathbb{Z})/P(\pi_1(X,*),\mathbb{Z})$ defined by $\Phi(F) = [F_*]$, where $[F_*]$ denotes the image of the crossed homomorphism F_* under the canonical projection to the quotient. To complete the proof of Theorem 4.2, it suffices to show that $\ker \Phi = E_1^- \cap [1]^-$.

Lemma 4.7. $E_1^- \cap [1]^- \subseteq \ker \Phi$.

Proof. We show that, if $F:(X_H,\tilde{*}) \to (S^1,1)$ is homotopic to the constant map at 1 through nontrivially equivariant maps, then

$$F_*([\gamma]) = \begin{cases} 0 & \text{if } [\gamma] \in H, \\ 2m & \text{if } [\gamma] \notin H, \end{cases}$$

where $m \in \mathbb{Z}$ is independent of $[\gamma]$. This says that $F_*([\gamma]) = m - [\gamma]\cdot m$ for all $[\gamma] \in \pi_1(X,*)$, so F_* is a principal homomorphism.

To establish this, let $\gamma: I \to X$ be a loop based at $*$, and let $\widetilde{\gamma}: I \to X_H$ be the unique lift of γ based at $\tilde{*}$. For notational convenience, let $x = \widetilde{\gamma}(1)$, so that either $x = \tilde{*}$ or $x = \alpha_H(\tilde{*})$. Denote by i_0 and i_1 the inclusions of X_H to the ends of $X_H \times I$, and by $\nu_x: I \to X_H \times I$ the path defined by $\nu_x(t) = (x,t)$. Noting that $i_0 \circ \widetilde{\gamma}$ is homotopic relative to its endpoints to $\nu_{\tilde{*}}(i_1 \circ \widetilde{\gamma})\nu_x{}^{-1}$, we obtain a map $K: I\times I \to X_H \times I$ satisfying:

(1) $K(-,0) = i_0 \circ \widetilde{\gamma}$,

(2) $K(-,1) = \nu_{\tilde{*}}(i_1 \circ \widetilde{\gamma})\nu_x{}^{-1}$,

(3) $K(0,t) = (\tilde{*}, 0)$,
(4) $K(1,t) = (x, 0)$.
Let $J : X_H \times I \to S^1$ be an equivariant null-homotopy of F, so that:
(1) $J(-,0) = F$,
(2) $J(-,1) = 1$,
(3) $J_t \circ \alpha_H = \tau \circ J_t$.
Now consider the composition $J \circ K : I \times I \to S^1$. It has the following properties:
(1) $(J \circ K)(-,0) = F \circ \widetilde{\gamma}$,
(2) $(J \circ K)(-,1) = (J \circ \nu_{\tilde{*}})(J \circ i_1 \circ \widetilde{\gamma})(J \circ \nu_x)^{-1}$,
(3) $(J \circ K)(0,t) = 1$,
(4) $(J \circ K)(1,t) = 1$.

Thus $[F \circ \widetilde{\gamma}] = [J \circ \nu_{\tilde{*}}][J \circ i_1 \circ \widetilde{\gamma}][J \circ \nu_x]^{-1} = [J \circ \nu_{\tilde{*}}][1][J \circ \nu_x]^{-1} = [J \circ \nu_{\tilde{*}}][J \circ \nu_x]^{-1}$ in $\pi_1(S^1, 1)$. Now examine two cases:

Case 1. $[\gamma] \in H$. In this case $\widetilde{\gamma}$ is a loop, so $x = \widetilde{\gamma}(1) = \tilde{*}$. Thus $[F \circ \widetilde{\gamma}] = [J \circ \nu_{\tilde{*}}][J \circ \nu_{\tilde{*}}]^{-1} = [\Gamma]^0$, so $F_*([\gamma]) = 0$.

Case 2. $[\gamma] \notin H$. In this case $\widetilde{\gamma}$ is a path from $\tilde{*}$ to $\alpha_H(\tilde{*})$, so $x = \alpha_H(\tilde{*})$. Because J is nontrivially equivariant, $J \circ \nu_x = J \circ \nu_{\alpha_H(\tilde{*})} = \tau \circ J \circ \nu_{\tilde{*}}$, so we have $[F \circ \widetilde{\gamma}] = [J \circ \nu_{\tilde{*}}][\tau \circ J \circ \nu_{\tilde{*}}]^{-1} = [J \circ \nu_{\tilde{*}}]\tau_*[\tau \circ J \circ \nu_{\tilde{*}}] = [J \circ \nu_{\tilde{*}}][J \circ \nu_{\tilde{*}}] = [\Gamma]^{2m}$, where $[J \circ \nu_{\tilde{*}}] = [\Gamma]^m$. Thus $F_*([\gamma]) = 2m$, and m is independent of $[\gamma]$.

This finishes the proof of Lemma 4.7. □

Lemma 4.8. $\ker \Phi \subseteq E_1^- \cap [1]^-$.

Proof. Suppose that $F : (X_H, \tilde{*}) \to (S^1, 1)$ is nontrivially equivariant and

$$F_* : \pi_1(X, *) \to \mathbb{Z}$$

is a principal homomorphism. We must show that F is homotopic to 1 through nontrivially equivariant maps.

A first observation is that, without loss of generality, we may assume that X and X_H are 1-dimensional complexes. For suppose we could homotope $F|_{X_H^{(1)}}$ to 1 through nontrivially equivariant maps. Then, by Proposition 4.1, we could obtain a new nontrivially equivariant map $F' : (X_H, \tilde{*}) \to (S^1, 1)$, equivariantly homotopic to F, satisfying $F'(X_H^{(1)}) = 1$. F' could then be equivariantly homotoped to 1 in the following manner: For a 2-cell $\widetilde{e}_i^2$ of X_H, $F'|_{\tilde{e}_i^2}$ represents an element of $\pi_2(S^1, 1)$. Since this group is trivial, $F'|_{\tilde{e}_i^2}$ can be homotoped to the constant map at 1. Use the "corresponding" homotopy to homotope $F'|_{\alpha_H(\tilde{e}_i^2)}$ to the constant map at 1. Do this for all such 2-cells $\widetilde{e}_i^2$ and $\alpha_H(\widetilde{e}_i^2)$. Extending this homotopy of $F'|_{X_H^{(2)}}$ using Proposition 4.1, we obtain a new nontrivially equivariant map $F'' : (X_H, \tilde{*}) \to (S^1, 1)$, still equivariantly homotopic to our original F, satisfying $F''(X_H^{(2)}) = 1$. Then continue in this fashion over all higher skeleta, using the fact that S^1 is aspherical.

Thus we will assume that X and X_H are 1-dimensional.

If $\widetilde{e}_i^1$ is a 1-cell in X_H with both of its ends attached at $\tilde{*}$, then a characteristic map for $\widetilde{e}_i^1$ is a loop in X_H based at $\tilde{*}$, so it is the lift of a loop $\gamma : I \to X$ based at $*$, and

$[\gamma] \in H$. Since F_* is assumed to be a principal homomorphism, $F_*([\gamma]) = 0$, meaning that $[F|_{\tilde{e}_i^1}]$ is trivial in $\pi_1(S^1, 1)$. Homotope $F|_{\tilde{e}_i^1}$ rel $\tilde{*}$ to the constant map at 1. Use the corresponding homotopy to homotope $F|_{\alpha_H(\tilde{e}_i^1)}$ rel $\alpha_H(\tilde{*})$ to the constant map at 1. Do this for all such 1-cells $\widetilde{e}_i^1$ and $\alpha_H(\widetilde{e}_i^1)$.

If $\widetilde{e}_j^1$ is a 1-cell in X_H attached at $\tilde{*}$ and $\alpha_H(\tilde{*})$, then $\alpha_H(\widetilde{e}_j^1)$ is also such a 1-cell. A characteristic map can be chosen for $\widetilde{e}_j^1$ which is the lift of a loop $\gamma : I \to X$ based at $*$, and $[\gamma] \notin H$. Since $F_*([\gamma]) = 2m$, $F|_{\tilde{e}_j^1}$ can be homotoped rel $\{\tilde{*}, \alpha_H(\tilde{*})\}$ to a path wrapping $2m$ times counter-clockwise around S^1 from 1 back to 1 as $\widetilde{e}_j^1$ is traversed from $\tilde{*}$ to $\alpha_H(\tilde{*})$. Use the corresponding homotopy to homotope $F|_{\alpha_H(\tilde{e}_j^1)}$ rel $\{\alpha_H(\tilde{*}), \tilde{*}\}$ to a path wrapping $2m$ times clockwise around S^1 from 1 back to 1 as $\alpha_H(\widetilde{e}_j^1)$ is traversed from $\alpha_H(\tilde{*})$ to $\tilde{*}$. Do this for all such 1-cells $\widetilde{e}_j^1$ and $\alpha_H(\widetilde{e}_j^1)$.

The result of this process is a new nontrivially equivariant map $F' : (X_H, \tilde{*}) \to (S^1, 1)$, equivariantly homotopic to F, which is now in a standard form. Although we will not attempt to write it down, the reader is invited to verify that there is a homotopy of F' to 1 through nontrivially equivariant maps. (Under this homotopy, the image of $\tilde{*}$ travels m times counter-clockwise around S^1, while the image of $\alpha_H(\tilde{*})$ travels the conjugate path m times clockwise around S^1, from 1 back to 1.) This proves Lemma 4.8. □

Theorem 4.2 is now proved. (We leave it to the interested reader to check that our isomorphism is natural in a suitable categorical sense.)

5. A Stallings' fibration theorem for semibundles

If M is a closed, connected, orientable and irreducible 3-manifold, and $f : M \to S^1$ is a fiber-bundle with fiber a connected surface Σ, then $f_* : \pi_1(M, *) \to \pi_1(S^1, 1)$ may be thought of as an element of $H^1(\pi_1(M, *); \mathbb{Z})$, where $\pi_1(M, *)$ acts trivially on $\mathbb{Z}$. By the long-exact homotopy sequence of the fibration $\Sigma \hookrightarrow M \to S^1$, f_* is a surjection and $\ker f_* = i_*\pi_1(\Sigma, *)$ is finitely-generated. In his fibration theorem (see [8], or Theorem 11.6 in [3]), Stallings shows that these conditions on a class in $H^1(\pi_1(M, *); \mathbb{Z})$, conditions which necessarily follow from the class being represented by a surface-bundle, are in fact sufficient conditions. That is, Stallings proves that, if a class in $H^1(\pi_1(M, *); \mathbb{Z})$ is represented by a surjection $\theta : \pi_1(M, *) \to \pi_1(S^1, 1)$ with finitely-generated kernel, then M is a fiber-bundle over S^1 with fiber a connected surface Σ, and $i_*\pi_1(\Sigma, *) = \ker \theta$.

In this section we prove an analogue of this result, a Stallings' fibration theorem for semibundles. Or, more correctly, we show that such a theorem has already been proved, by Hempel and Jaco, although their result is not phrased in the unifying language of cohomology.

We begin by briefly summarizing what we have already done. In Section 1 we defined what it means for a map $f : M \to D^1$ to be an H-semibundle with regular fiber Σ, where H is a halving of $\pi_1(M, *)$ and Σ is a connected surface. In Section 2 we defined the twisted cofundamental group, $\overline{\pi}^1(X/\alpha)$, of an orbit space X/α, and we explained how an H-semibundle $f: M \to D^1$ gives rise to a pair of classes in $\bar{\pi}^1(M_H/\alpha_H)$. In Section 3 we discussed group cohomology—in particular, the group $H^1(\Pi;\mathbb{Z})$, the first cohomology group of Π with twisted integer coefficients, where (Π, H) acts semitrivially on $\mathbb{Z}$. In Section 4 we established a natural isomorphism between $\bar{\pi}^1(X_H/\alpha_H)$ and the twisted cohomology group $H^1(\pi_1(X,*);\mathbb{Z})$.

Now let us put these pieces together. From Corollary 3.3 and the isomorphism of Theorem 4.2, the twisted cofundamental group of a compact 3-manifold is a finitely-generated abelian group with a torsion subgroup isomorphic to $\mathbb{Z}_2$. Also by Theorem 4.2, an H-semibundle $f: M \to D^1$ gives rise to a pair of classes in the twisted cohomology group $H^1(\pi_1(M,*);\mathbb{Z})$, specifically $[F_*]$ and $[(\tau \circ F)_*] = -[F_*]$, where $F: M_H \to S^1$ is a nontrivially equivariant surface-bundle covering f.

What can we say about the crossed homomorphism $F_*: \pi_1(M,*) \to \mathbb{Z}$ representing a semibundle class in the twisted cohomology group $H^1(\pi_1(M,*);\mathbb{Z})$? We have

Proposition 5.1. *Suppose $f: M \to D^1$ is an H-semibundle with regular fiber Σ, and $F: M_H \to S^1$ is a nontrivially equivariant surface-bundle covering f. Then the crossed homomorphism $F_*: \pi_1(M,*) \to \mathbb{Z}$ has the following properties*:

(1) *$F_*|_H$ is a surjective homomorphism,*

(2) *$\ker(F_*|_H)$ is finitely-generated.*

Proof. (1) By its definition in Lemma 4.4, we have $F_*|_H \circ (Q_H)_* = F_*$, where $Q_H: M_H \to M$ is the two-sheeted covering, and the F_* on the right-hand side of the equality is the ordinary homomorphism $F_*: \pi_1(M_H, \tilde{*}) \to \pi_1(S^1, 1)$ induced by $F: M_H \to S^1$. We identify $\mathbb{Z}$ and $\pi_1(S^1,1)$ as in Lemma 4.4. Since $F: M_H \to S^1$ is a surface-bundle with connected fiber, $F_*: \pi_1(M_H,\tilde{*}) \to \pi_1(S^1,1)$ is a surjection, which implies that $F_*|_H$ is a surjection. $F_*|_H$ is an ordinary homomorphism because the action of $\pi_1(M,*)$ on $\mathbb{Z}$ is trivial on H.

(2) Since $F_*|_H \circ (Q_H)_* = F_*$, $\ker(F_*|_H) = (Q_H)_*(\ker F_*)$. But $(Q_H)_*$ is an injection and $\ker F_* = i_*\pi_1(\Sigma, \tilde{*})$ is finitely-generated.

This proves Proposition 5.1. □

We conclude by showing that the necessary conditions given in Proposition 5.1 are, in fact, sufficient algebraic conditions to guarantee that M is a semibundle. Specifically, we have

Theorem 5.2. *Suppose M is a closed, connected, orientable and irreducible 3-manifold, and let H be a halving of M. Let $[\theta] \in H^1(\pi_1(M,*);\mathbb{Z})$ be a twisted cohomology class, and suppose that the crossed homomorphism $\theta: \pi_1(M,*) \to \mathbb{Z}$ satisfies*:

(1) *$\theta|_H$ is a surjective homomorphism,*

(2) *$\ker(\theta|_H)$ is finitely-generated.*

Then $M = M_1 \cup M_2$, where $M_1 \cap M_2 = \Sigma$ is a properly-embedded, two-sided, incompressible surface, M_i is a twisted I-bundle over some surface with Σ the corresponding 0-sphere bundle, and $\ker(\theta|_H) = i_*\pi_1(\Sigma,*)$.

Proof. Before beginning the proof, one comment is in order. Since M is assumed to be irreducible, every S^2 embedded in M bounds a D^3 embedded in M. In particular, M does not contain a fake 3-cell. This is our way of avoiding any complications which might arise from the unresolved status of the Poincaré conjecture.

By work of Hempel and Jaco (see [4], or Theorem 11.8 in [3]), it suffices to construct a surjection $\hat{\theta}:\pi_1(M,*) \to \mathbb{Z}_2 * \mathbb{Z}_2$ with $\ker\hat{\theta} = \ker(\theta|_H)$. As is well known, $\mathbb{Z}_2 * \mathbb{Z}_2$ is isomorphic to the infinite dihedral group $D = \{a, b\colon\ bab = a^{-1},\ b^2 = 1\}$. We will view D as the semidirect product $\mathbb{Z} \rtimes (\pi_1(M,*)/H)$, where the generator of $\pi_1(M,*)/H \approx \mathbb{Z}_2$ acts on $\mathbb{Z}$ by negation. Specifically, $D = \{(m, \overline{[\gamma]})\colon\ m \in \mathbb{Z},\ \overline{[\gamma]} = [\gamma]H \in \pi_1(M,*)/H\}$, with multiplication defined by $(m, \overline{[\gamma]})\cdot(m', \overline{[\gamma']}) = (m + \overline{[\gamma]}\cdot m', \overline{[\gamma]}\,\overline{[\gamma']})$, where $\overline{[\gamma]}\cdot m' = m'$ if $[\gamma] \in H$ and $\overline{[\gamma]}\cdot m' = -m'$ if $[\gamma] \notin H$.

Define $\hat{\theta}:\pi_1(M,*) \to D$ by $\hat{\theta}([\gamma]) = (\theta([\gamma]), \overline{[\gamma]})$. Because $\theta:\pi_1(M,*) \to \mathbb{Z}$ is a crossed homomorphism, $\hat{\theta}$ is an ordinary homomorphism, and $\ker\hat{\theta} = \ker\theta \cap H = \ker(\theta|_H)$. To see that $\hat{\theta}$ is surjective, it suffices to show that generators of D lie in the image of $\hat{\theta}$. Because $\theta|_H$ is a surjection, $(1, 0)$ lies in the image of $\hat{\theta}$. To finish the proof, we must produce $[\eta] \in \pi_1(M,*) - H$ with $\theta([\eta]) = 0$. Pick any $[\mu] \in \pi_1(M,*) - H$. Since $\theta|_H$ is surjective, there exists $[\gamma] \in H$ with $\theta([\gamma]) = \theta(\mu)$. Then $\theta([\mu][\gamma]) = \theta([\mu]) + [\mu]\cdot\theta([\gamma]) = \theta([\mu]) - \theta([\gamma]) = 0$, so let $[\eta] = [\mu][\gamma]$. □

Acknowledgements

The author would like to take this opportunity to thank his thesis advisor, Allen Hatcher, for introducing him to semibundles, for suggesting that they might be amenable to study using "twisted homology", and for many valuable mathematical discussions.

References

[1] K. Borsuk, Sur les groupes des classes de transformations continues, C. R. Acad. Sci. Paris 202 (1949) 1400–1403.
[2] A. Hatcher, Notes on basic 3-manifold topology, Preliminary version (1989).
[3] J. Hempel, 3-Manifolds, Ann. of Math. Stud. 86 (Princeton Univ. Press, Princeton, NJ, 1976).
[4] J. Hempel and W. Jaco, Fundamental groups of 3-manifolds which are extensions, Ann. of Math. 95 (1972) 86–98.
[5] S. Hu, Homotopy Theory (Academic Press, New York, 1959).
[6] S. Illman, Equivariant algebraic topology, Ph.D. Thesis, Princeton Univ., Princeton, NJ (1972).
[7] E.H. Spanier, Borsuk's cohomotopy groups, Ann. of Math. 50 (1949) 203–245.
[8] J. Stallings, On fibering certain 3-manifolds, in: M.K. Fort Jr, ed., Topology of 3-Manifolds and Related Topics (Prentice-Hall, Englewood Cliffs, NJ, 1962) 95–100.
[9] W.P. Thurston, A norm on the homology of 3-manifolds, Mem. Amer. Math. Soc. 339 (1986).
[10] G.W. Whitehead, Elements of Homotopy Theory, Graduate Texts in Math. 61 (Springer, New York, 1978).